The Finite Groups of Birational Transformations of a Net of Cubics.

By Lewis C. Cox.

Introduction.

1. The classification of non-linear periodic birational transformations in the plane into finite groups is due to S. Kantor * and A. Wiman.† The former showed that all periodic birational transformations in a plane can be transformed by combinations of quadratic transformations into a finite number of types having at most eight fundamental points. In the case of seven fundamental points, he determined the different transformations, but in their groups he made errors which A. Wiman subsequently corrected.

The method of Wiman was dependent upon the fact that these groups are isomorphic with the groups of transformations of the bitangents of a plane quartic C_4 which were known.‡

The object of this paper is to establish a method which enables one to determine the Cremona transformations with seven fundamental points which correspond to a given linear transformation of the quartic curve.

The method used is to first find the twenty-eight bitangents of the quartic. Hence a cubic surface is determined, having the quartic as a plane section of a particular cone which is tangent to the surface and has its vertex on the surface. This cubic surface is then depicted upon the plane of the seven fundamental points. The space transformations leaving the surface invariant and corresponding to a collineation of the quartic are next found. The corre-

* S. Kantor, "Premiers Fondaments Pour Une Theorie Des Transformations Periodiques Univoques," *Atti della Reale Accademia delle scienze fisiche e matematiche di Napoli*, Series 2, Vol. III, No. 7 and Vol. IV, No. 2 (1891), pp. 1–335.

† A. Wiman, "Ueber die Endlichen Gruppen von eindeutigen Transformationen in der Ebene," *Mathematische Annalen*, Vol. XLVIII (1896); pp. 194–240.

‡ R. de Paolis, "Le trasformazioni piane doppie," *Atti delle Reale Accademia di Lincei*, Series 3a, Vol. I (1877), pp. 511–544. "Le trasformazioni piane doppia di secondo ordine, e la sua applicazione alla geometria non euclidea," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 31–50. "La trasformazione piana doppia di terzo ordine, primo genere, e la sua applicazione alle curve del quarto ordine," *Atti d. R. Accad. di Lincei*, Series 3a, Vol. II (1878), pp. 851–878.

sponding transformation on the plane of the seven fundamental points can then be determined. The method of determination is first developed for any collineation and is given in full for the cases in which the quartic is invariant under the groups of collineations G_{24}, G_{96}, G_{168}.

General Case.

2. Consider the cubic surface

$$F_3 \equiv f_1(x, y, z)u^2 + 2f_2(x, y, z)u + f_3(x, y, z) = 0, \tag{1}$$

in which $f_i(x, y, z)$ is a ternary form of order i. Equation (1) can be arranged in the form

$$\{f_1(x, y, z)u + f_2(x, y, z)\}^2 = f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z), \tag{2}$$

which represents a composite surface made up of F_3 and the plane $f_1(x, y, z) = 0$, the latter being tangent to F_3 at the point $0 \equiv (0, 0, 0, 1)$. Solving for u we get

$$u = \frac{-f_2(x, y, z) \pm \sqrt{f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z)}}{f_1(x, y, z)}. \tag{3}$$

Hence, any line

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$$

through the point $0 \equiv (0, 0, 0, 1)$ on the surface meets F_3 in two other points. These two residual points will coincide if

$$f_2^2(\lambda, \mu, \nu) - f_1(\lambda, \mu, \nu)f_3(\lambda, \mu, \nu) = 0,$$

that is, the line is tangent to the surface and passes simply through 0.

Therefore,

$$K_4 \equiv f_2^2(x, y, z) - f_1(x, y, z)f_3(x, y, z) = 0, \tag{4}$$

is the tangent cone with its vertex at 0. The plane $u = 0$ upon which we project $F_3 = 0$ is a double plane because the residual intersections of the lines through 0 with the surface $F_3 = 0$ project into single points. The cone $K_4 = 0$ intersects the plane $u = 0$ in the quartic curve

$$C_4 \equiv \begin{Bmatrix} f_2^2 - f_1 f_3 = 0, \\ u = 0. \end{Bmatrix} \tag{5}$$

From the form of C_4 it is seen that $\begin{Bmatrix} f_1(x, y, z) = 0 \\ u = 0 \end{Bmatrix}$ is a bitangent to C_4.

The Lines on F_3.

3. Let $\phi_1(x, y, z) = 0$ be the equation of the plane through 0 and one of the other twenty-seven bitangents to C_4. If one of the variables is eliminated

between $\left\{\begin{matrix}\phi_1=0\\ u\;=0\end{matrix}\right\}$ and (5) the remaining binary form must be a perfect square, whose roots determine the coordinates of the points of contact. Hence, eliminating z between

$$\phi_1(x, y, z)=0 \quad \text{and} \quad (f_1u+f_2)^2=f_2^2-f_1f_3$$

we obtain an expression that is rationally factorable:

$$[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0, \tag{6}$$

in which F_1, F_2 and ϕ_2 are binary forms in x and y. Since equation (2) represents a composite surface whose components are F_3 and $f_1=0$, the equations

$$\left\{\begin{matrix}[F_1u+F_2+\phi_2]\,[F_1u+F_2-\phi_2]=0\\ \phi_1(x, y, z) \qquad =0\end{matrix}\right\} \tag{7}$$

represent the curves $\left\{\begin{matrix}\phi_1(x, y, z)=0\\ F_3 \quad =0\end{matrix}\right\}$ and $\left\{\begin{matrix}f_1(x, y, z)=0\\ \phi_1(x, y, z)=0\end{matrix}\right\}$, the latter defining an extraneous line not lying on F_3. Eliminating the factor f_1 from the factorable equation in (7) we find a remaining linear factor. This taken simultaneously with $\phi_1(x, y, z)=0$ fixes a line of F_3. The residual section of $\left\{\begin{matrix}\phi_1(x, y, z)=0\\ F_3 \quad =0\end{matrix}\right\}$ is a proper conic section.

Therefore, we conclude that the planes through 0 and the twenty-seven bitangents to C_4 meet F_3 in the twenty-seven lines on F_3 and in twenty-seven residual conic sections. Hence each part of the composite curve, consisting of a line and its residual conic which lie not only on F_3 but also in the plane through 0 and one of the twenty-seven bitangents, corresponds to one of the twenty-seven bitangents to C_4.

The plane $f_1(x, y, z)=0$ is tangent to F_3 at 0. It intersects F_3 in the cubic curve

$$\left\{\begin{matrix}f_1(x, y, z)=0,\\ 2f_2(x, y, z)u+f_3(x, y, z)=0.\end{matrix}\right\} \tag{8}$$

It also intersects the plane $u=0$ in

$$\left\{\begin{matrix}f_1(x, y, z)=0,\\ u \quad =0,\end{matrix}\right\}$$

which is the twenty-eighth bitangent to C_4. Therefore, the point 0 and the cubic curve (8) correspond to the twenty-eighth bitangent to C_4.

The question arises, which sign in the second member of the equation

$$F_1u+F_2=\pm\phi_2 \tag{9}$$

corresponds to the cubic curve. Consider the equation

$$uf_1+f_2=+\sqrt{f_2^2-f_1f_3} \tag{10}$$

which is found from (2). Substitute the coordinates of a point $P \equiv (x, y, z, u)$ which lies on the cubic (8). The equation (10) is satisfied. Hence, the plus sign in the second member of (9) corresponds to the cubic (8). The point $(0, 0, 0, 1)$ is the only point satisfying

$$uf_1 + f_2 = -\sqrt{f_2^2 - f_1 f_3}.$$

Hence, the minus sign corresponds to the point 0.

Correspondence of Transformations.

4. When C_4 is of genus 3, the only transformations which leave it invariant are collineations. The quartic curves admitting groups of such transformations are enumerated by S. Kantor.*

Let T represent any linear transformation which leaves C_4 invariant. It will permute the bitangents among themselves. From equation (2) we find

$$f_1(x, y, z)u + f_2(x, y, z) = \pm \{ f_1(x', y', z')u' + f_2(x', y', z') \}. \tag{11}$$

The relations (11) and T define two space transformations Q', Q''. Any plane through 0 and a bitangent to C_4 is changed by Q' or Q'' into a plane through 0 and some bitangent. Hence, the lines in which these planes meet F_3 are either interchanged, in which case the space transformation on F_3 is linear, or else each line is changed into the residual conic in the plane of the second line and the transformation is quadratic.

If we now use the Grassman depiction of the cubic surface F_3 upon a second plane $w = 0$, the plane sections through 0 are transformed into the ∞^2 cubics through the image of 0 and having for additional basis points the images of the six special lines on F_3 which are depicted as the fundamental points. The point 0, the cubic curve (8) and the composite curves lying on F_3 in the twenty-seven planes through 0 and the lines of the surface are depicted in the plane $w = 0$ as the complete fundamental system of the required Cremona transformation.

Since Q', Q'' interchange the twenty-eight special sections of F_3, one changing a line into a line, the other changing a line into a conic, the corresponding transformations T', T'' interchange the parts of the fundamental system. Hence, from T we can derive two birational Cremona transformations.

The Geiser Transformation.

5. In considering $T \equiv \begin{pmatrix} x, & y, & z \\ x', & y', & z' \end{pmatrix}$ in the plane $u = 0$ we are led to two

* S. Kantor, "Theorie der Endlichen Gruppen von eindeutigen Transformationen in der Ebene," Berlin (1895); 120 pages, see p. 86.

space transformations defined by the equations of T and (11). They are:

$$I \equiv \begin{pmatrix} x, & y, & z, & u \\ x', & y', & z', & u' \end{pmatrix},$$

and

$$\Gamma = \begin{cases} x = x' \\ y = y' \\ z = z' \\ w = -\{u' f_1(x', y', z') + f_2(x', y', z')\} - f_2(x', y', z'). \end{cases}$$

The former is the identical transformation.

The latter is a non-linear transformation possessing the following properties. It leaves all the planes and lines through 0 invariant, but interchanges the points P' and P'' which lie on F_3 and are collinear with 0. Therefore, it changes any line on F_3 into the residual conic lying on F_3 and coplanar with 0 and the line. Hence, the transformation is quadratic. The transformation Γ leaves invariant the locus of points R on F_3 which are fixed by the condition

$$f_2^2(\lambda, \mu, \nu) - f_1(\lambda, \mu, \nu) f_3(\lambda, \mu, \nu) = 0.$$

The cone $K_4 = 0$ and the surface $F_3 = 0$ have contact along a sextic space curve C_6 which is invariant point for point under the transformations I and Γ.

Hence, the transformation θ_2 in the plane $w = 0$ corresponding to Γ has the following properties: It interchanges the images of P' and P''. Therefore θ_2 is involutorial. It leaves invariant as a whole every cubic curve through the seven fundamental points. It leaves invariant point for point the image of C_6, which is a plane sextic curve Γ_6 having a double point at each of the seven fundamental points. It is of genus 3, and is non-hyperelliptic.* The transformation θ_2 interchanges the parts of every composite cubic passing through the seven fundamental points. A line through two such points and the conic through the remaining five interchange. Each fundamental point goes into a rational cubic having its double point whose image is the point. This cubic passes singly through the remaining six fundamental points. Each such fundamental cubic and its associated point are interchanged by θ_2. Hence, the transformation is the Geiser involutorial transformation.†

* V. Snyder, "On a Special Algebraic Curve Having a Net of Minimum Adjoint Curves," *Bulletin of the American Mathematical Society*, Vol. XIV (1907), pp. 70–74.

† R. Sturm, "Die Lehre von den Geometrischen Verwandtschaften," Vol. IV, Leipzig (1908), pp. 95–98. The equations of the Geiser transformation are given by V. Snyder. "An Application of the (1, 2) Quaternary Correspondence to the Weddle and Kummer Surfaces," *Transactions of the American Mathematical Society*, Vol. II (1911), pp. 354–366.

6. Let $T_1, T_2, T_3, \ldots, T_n$ be a set of operations generating the group G of the quartic C_4. Let $Q'_1, Q'_2, \ldots, Q'_m$ be their corresponding space transformations. Let $T'_1, T'_2, T'_3, \ldots, T'_n$ be their corresponding Cremona transformations in the plane $w=0$. Let the symbol A, B represent a group generated by A and B. Let the symbol (r, s) be read "possesses an r to s isomorphism with."

Subgroups of G may be found

$$[T'_1, \theta_2]\,(1, 1)\,[Q'_1, \Gamma]\,(2, 1)\,[T_1],$$
$$[T'_1, T'_j, \theta_2]\,(1, 1)\,[Q'_1, Q'_j, \Gamma]\,(2, 1)\,[T_1, T_j],$$
$$[T'_1, T'_j, T'_k, \theta_2]\,(1, 1)\,[Q'_2, Q'_j, Q'_k, \Gamma]\,(2, 1)\,[T_1, T_j, T_k].$$

Similarly other subgroups may be found.

Determination of G_{48}.

7. The plane quartic curve

$$C_4 \equiv \begin{cases} x^4+y^4+z^4+k(x^2y^2+x^2z^2+y^2z^2)=0 \\ u=0 \end{cases} \tag{13}$$

is invariant under a group $\overline{G}_{24}$ of linear transformations of order 24.* It is generated by the permutation $\overline{G}_6$ on the three letters x, y, z and the harmonic homology defined by changing the sign of any one of the variables.

Bitangents to C_4.

8. Certain lines in the plane $u=0$ are bitangent to C_4. In order to find them it is necessary to identify four special types. The others are found from the collineations of $\overline{G}_6$. We wish to determine, if possible, a relation between k and μ, such that $y=\mu x$ shall be a bitangent; similarly, a relation between k and λ such that $z=\pm\lambda(x\pm y)$ shall be a bitangent.

Eliminate y from $\begin{Bmatrix} y=\pm\mu x \\ u=0 \end{Bmatrix}$ and the equation of C_4 and we obtain

$$\begin{cases} x^4+k(1+\mu^2)x^2z^2+(1+k\mu^2+\mu^4)z^4=0, \\ u=0. \end{cases} \tag{14}$$

The line is a bitangent when the first member of (14) is a square. The condition for this is

$$k^2(1+\mu^2)^2-4k\mu^2-4k(1+\mu^4)=0. \tag{15}$$

Solve for k and select that value for which the first member of (13) is not a perfect square. We get

$$k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \tag{16}$$

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* E. Ciani, "Contributo alla teoria del gruppo di 168 collineationi piane," *Annali di Mathematica*, Series 3, Vol. V (1901), pp. 35–55. See p. 43.

Hence $y=\pm\mu x$ must be a bitangent of the C_4, (13) having the value of k defined by (16), where μ may have any finite value.

Similarly, if we eliminate z between (13) and $z=\pm\lambda(x\pm y)$, place the discriminant of the resulting equation equal to zero, and factor out $(k-2)$ we obtain

$$(k+1)\lambda^4-k\lambda^2-1=0. \tag{17}$$

Solving for λ^2 we find $\lambda^2=+1$ or $\lambda^2=-\dfrac{1}{k+1}$. By substituting the second value of k in (16) we get

$$\lambda=\pm\frac{1+\mu^2}{1-\mu^2}. \tag{18}$$

Hence, the line $\begin{Bmatrix} z=\pm\lambda(x\pm y) \\ u=0 \end{Bmatrix}$ must be bitangent to C_4 for the special values of λ.

By operating on the bitangents with $\overline{G}_6$ we obtain the following bitangents:

(1) $y=-\mu z$,	(14) $y=\mu z$,
(2) $z=-\mu x$,	(15) $z=\mu x$,
(3) $x=-\mu y$,	(16) $x=\mu y$,
(4) $x=\lambda(z-y)$,	(17) $x=\lambda(y-z)$,
(5) $y=\lambda(x-z)$,	(18) $y=\lambda(z-x)$,
(6) $z=\lambda(y-x)$,	(19) $z=\lambda(x-y)$,

(7) $x+y+z=0$,

(8) $x=-\lambda(y+z)$,	(20) $x=\lambda(y+z)$,
(9) $y=-\lambda(z+x)$,	(21) $y=\lambda(x+z)$,
(10) $z=-\lambda(x+y)$,	(22) $z=\lambda(x+y)$,
(11) $x=-\mu z$,	(23) $x=\mu z$,
(12) $y=-\mu x$,	(24) $y=\mu x$,
(13) $z=-\mu y$,	(25) $z=\mu y$,

(26) $-x+y+z=0$,
(27) $x-y+z=0$,
(28) $x+y-z=0$,

The Cubic Surface.

9. The next step is to construct the cubic surface having the properties explained in the general case. Select the four bitangents

$$\begin{Bmatrix} x+y+z=0 \\ u\quad=0 \end{Bmatrix},\quad \begin{Bmatrix} -x+y+z=0 \\ u\quad=0 \end{Bmatrix},\quad \begin{Bmatrix} x-y+z=0 \\ u\quad=0 \end{Bmatrix},\quad \begin{Bmatrix} x+y-z=0 \\ u\quad=0 \end{Bmatrix}.$$

The planes passing through $0 \equiv (0, 0, 0, 1)$ and these special lines are evidently

$$x+y+z=0, \quad x-y+z=0,$$
$$-x+y+z=0, \quad x+y-z=0.$$

Take $x+y+z=0$ to be the plane tangent to F_3 at 0. The cubic surface has an equation of the form

$$F_3 \equiv (x+y+z)u^2+2p(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)=0. \quad (19)$$

The tangent cone at $(0, 0, 0, 1)$ has the equation

$$(p^2+1)\Sigma x^4+2(p^2-1)\Sigma x^2y^2=0. \quad (20)$$

Comparing the latter with the first one of the two equations in (13) we notice that

$$\frac{2(p^2-1)}{p^2+1}=k=\frac{4\mu^2}{(1+\mu^2)^2}-2. \quad (21)$$

Hence,

$$p^2=\frac{2+k}{2-k} \quad \text{or} \quad p=\frac{\pm\mu}{\sqrt{1+\mu^2+\mu^4}}.$$

Either sign may be used for p. The cubic surface

$$F_3 \equiv (x+y+z)u^2+\frac{2\mu(x^2+y^2+z^2)u}{\sqrt{1+\mu^2+\mu^4}}+(-x+y+z)(x-y+z)(x+y-z)=0 \quad (22)$$

is selected for our discussion. Rearranging (19) we find

$$(x+y+z)u+\frac{2\mu(x^2+y^2+z^2)}{\sqrt{1+\mu^2+\mu^4}}=\pm(p^2+1)\sqrt{\Sigma x^4+\frac{2p^2-1}{p^2+1}\Sigma x^2y^2} \quad (23)$$

which by means of (21) can be reduced to the form

$$\{(x+y+z)v+\mu(x^2+y^2+z^2)\}^2=(1+\mu^2)\{\Sigma x^4+k\Sigma x^2y^2\}, \quad (24)$$

where $v=u\sqrt{1+\mu^2+\mu^4}$ and $k=\dfrac{4\mu^2}{(1+\mu^2)^2}-2$.

Equation (24) defines the composite surface F_3 and $(x+y+z)=0$.

The Lines on the Cubic Surface.

10. Each plane determined by 0 and some one of the twenty-seven bitangents to the quartic C_4 is a bitangent plane to F_3. The plane $x+y+z=0$, containing a bitangent to C_4, is tangent to F_3 at 0. It meets F_3 in a non-composite cubic curve.

We find each of the 27 lines upon F_3, together with their residual conics in the plane through 0 by solving their equations simultaneously with the equation of the composite surface (24).

The latter equation is reduced to the form

$$(x+y+z)v+\mu(x^2+y^2+z^2)=\pm(1+\mu^2)\cdot\phi_2, \tag{25}$$

where ϕ_2 corresponds to ϕ_2 in equation (6). One sign in the right member corresponds to a line on F_3 when one of the twenty-seven special planes is taken simultaneously with it. The opposite sign corresponds to a conic section lying in the bitangent plane used.

Equation (7) needs the same consideration as was given to the plane $f_1(x, y, z)=0$ in Section 3. We may show by the same reasoning as was employed there that the positive sign in the right member of (25) corresponds to the cubic curve

$$\left.\begin{aligned} 2\mu(x^2+y^2+z^2)u+(-x+y+z)(x-y+z)(x+y-z)&=0,\\ x+y+z&=0.\end{aligned}\right\} \tag{26}$$

The following list of equations represents the lines and the residual conic sections unless otherwise stated. The equations are numbered to correspond with the bitangents given previously. The first two equations in each trio fix a line, the first and third fix the residual conic sections in the plane connecting the line with 0. The third equation is added for a purpose which will be explained later.

(1)
$$\begin{gathered} y+\mu z=0\\ v+(1+\mu+\mu^2)(x-\overline{1-\mu}\cdot z)=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+\{(1+\mu^2)x^2-(1+\mu^4)z^2\}\end{gathered}$$

(4)
$$\begin{gathered} x=\lambda(z-y)\\ v+\mu\{(1-\lambda)z+(1+\lambda)y\}=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =-2\mu\cdot\lambda^2\left\{z^2+y^2-\frac{2\lambda^2+1}{\lambda^2}\cdot yz\right\}\end{gathered}$$

(7)
$$\begin{gathered} x+y+z=0 \text{ corresponds to } 0\equiv(0,0,0,1)\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+2\mu(x^2+z^2+xz) \text{ corresponds to the cubic curve}\\ \text{in the tangent plane.}\end{gathered}$$

(8)
$$\begin{gathered} x=-\lambda(y+z)\\ (1-\lambda)v+\mu(1+3\mu^2)(y+z)=0\\ (x+y+z)v+\mu(x^2+y^2+z^2)\\ =+2\mu\cdot\lambda^2\left\{y^2+z^2+\frac{2\lambda^2+1}{\lambda^2}yz\right\}\end{gathered}$$

(26)
$$\begin{aligned}
-x+y+z&=0\\
v&=0\\
(x+y+z)v+\mu(x^2+y^2+z^2)&\\
=-2\mu\{y^2+z^2+yz\}&
\end{aligned}$$

(10)
$$\begin{aligned}
z&=-\lambda(x+y)\\
(1-\lambda)v+\mu(1+3\mu^2)(x+y)&=0\\
(x+y+z)v+\mu(x^2+y^2+z^2)&\\
=+2\mu\cdot\lambda^2\cdot\left\{x^2+y^2+\frac{2\lambda^2+1}{\lambda^2}xy\right\}&
\end{aligned}$$

(11)
$$\begin{aligned}
x&=-\mu z\\
v+(1+\mu+\mu^2)(y-\overline{1-\mu}\,z)&=0\\
(x+y+z)v+\mu(x^2+y^2+z^2)&\\
=+\{(1+\mu^2)y^2-(1+\mu^4)z^2\}&
\end{aligned}$$

(12)
$$\begin{aligned}
y&=-\mu x\\
v+(1+\mu+\mu^2)(z-\overline{1-\mu}\,x)&=0\\
(x+y+z)v+\mu(x^2+y^2+z^2)&\\
=+\{(1+\mu^2)z^2-(1+\mu^4)x^2\}.&
\end{aligned}$$

Apply $\begin{pmatrix} x\,y\,z \\ y\,z\,x \end{pmatrix}$ to (1), (2), (4), (5), (8), (9) to get (2), (3), (5), (6), (9), (10). Write $-\mu$ for μ, $-v$ for v in (1), (2), (3) to get (14), (15), (16) and $-\lambda$ for λ in (4), (5), (6), (8), (9), (10) to get (17), (18), (19), (20), (21), (22). Interchange x and y in (3), (16), (26) to get (12), (24), (27); y and z in (1), (14), (27) to get (13), (25), (28); z and x in (2), (15) to get (11), (23).

Arrangement of the 27 Lines on F_3.

11. A well-known property of a non-singular cubic surface F_3 is that each line on F_3 is contained in a set of five planes each of which contains two other lines on F_3. By testing the equations in Section 10, we find that the line (26) forms a triangle with each pair of lines [(14), (25)], [(20), (8)], [(17), (4)], [(1), (13)], [(28), (27)] and, similarly, (28) with [(16), (24)], [(22), (10)], [(19), (6)], [(3), (12)], [(26), (27)], and (27) with [(15), (23)], [(21), (9)], [(18), (5)], [(2), (11)], [(28), (26)].

If we call $(26)\equiv C_{14}$* and $(16)\equiv C_{12}$, then (27) is a c line which we shall designate by c_{25}; (24) by c_{45} and (28) by c_{36}.

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* Schläfli's notation has been adopted. F. Schläfli, "An Attempt to Determine the Twenty-seven Lines upon a Surface of the Third Order and to Divide Such Surfaces into Species in Reference to the Reality of the Lines upon the Surface," *Quarterly Journal*, Vol. II (1858), pp. 55–65 and 110–120; see p. 115. K. Doehlemann, "Geometrische Transformationen," Teil II (1907), pp. 303–305.

The only four lines skew to (26) and (24) but intersecting (16) and (27) are (21), (23), (2) and (18), which must consist of two c lines, one a and one b line. Call them c_{34}, c_{46}, a_2, b_2, respectively. The only four lines skew to (16) and (27), but intersecting (26) and (24) are (14), (20), (4) and (13). Of these (14) is skew (28) and, therefore, must be c_{23}. Similarly, (20), (4) and (13) must be c_{26}, a_4, b_4, respectively. We see now that $(25)\equiv c_{56}$, $(8)\equiv c_{35}$, $(17)\equiv b_1$, $(1)\equiv a_1$, $(15)\equiv c_{13}$, $(9)\equiv c_{16}$, $(5)\equiv a_5$, $(11)\equiv b_5$, leaving unlettered the lines [(22), (10)], [(19), (6)], [(3), (12)].

Consider the double six

$$\begin{matrix} c_{14}, & c_{45}, & c_{34}, & c_{24}, & a_6, & b_6, \\ c_{16}, & c_{56}, & c_{36}, & c_{26}, & a_4, & b_4, \end{matrix}$$

(25) meets (10), (6) and (12) which must be c_{24}, a_6, b_6 in the same order, but (20) is skew to (10), (6) to (4), (12) to (13); hence, the order is correct. This completely determines the lettering of the lines which is as follows:

$(1)\equiv a_1$	$(8)\equiv c_{35}$	$(15)\equiv c_{13}$	$(22)\equiv c_{15}$
$(2)\equiv a_2$	$(9)\equiv c_{16}$	$(16)\equiv c_{12}$	$(23)\equiv c_{46}$
$(3)\equiv a_3$	$(10)\equiv c_{24}$	$(17)\equiv b_1$	$(24)\equiv c_{45}$
$(4)\equiv a_4$	$(11)\equiv b_5$	$(18)\equiv b_2$	$(25)\equiv c_{56}$
$(5)\equiv a_5$	$(12)\equiv b_6$	$(19)\equiv b_3$	$(26)\equiv c_{14}$
$(6)\equiv a_6$	$(13)\equiv b_4$	$(20)\equiv c_{26}$	$(27)\equiv c_{25}$
	$(14)\equiv c_{23}$	$(21)\equiv c_{34}$	$(28)\equiv c_{36}$

Instead of choosing (26) as a c line we might have taken it for an a_1 or b_1 line in which cases (16) would be b_1 or a_1, respectively. Hence, we may name the lines on F_3 in a variety of ways.

Identification of the Generators of G_{45}.

12. Select a transformation $T_3\equiv\begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ which leaves C_4 invariant. (The subscript in T_3 denotes the period.) Choose the transformation Q_3' which is defined by the equations

$$\begin{cases} x = y' \\ y = z' \\ z = x' \\ v = \dfrac{-\mu(x'^2+y'^2+z'^2)+(x'+y'+z')v'+\mu(x^2+y^2+z^2)}{x+y+z} \end{cases}$$

which in our case reduces to the linear space transformation

$$Q_3' = \begin{pmatrix} x, & y, & z, & v \\ y', & z', & x', & v' \end{pmatrix}.$$

It is one of the two space transformations which correspond to T_3. The transformation Q_3' permutes the twenty-eight planes through 0 among themselves. It leaves invariant the left member of each second degree equation listed in Section 10. It may or may not change the right member. The same is true of those second degree equations from which those planes are found, which with the bitangent planes fix the lines on F_3. The transformation Q_3' is operated as follows: Take for example the line a_1. The transformation $T_3 \equiv \begin{pmatrix} x, & y, & z \\ y', & z', & x' \end{pmatrix}$ transforms the bitangent $\begin{Bmatrix} y = -\mu z \\ u = \quad 0 \end{Bmatrix}$ into the bitangent $\begin{Bmatrix} z = -\mu x \\ u = \quad 0 \end{Bmatrix}$; hence, Q_3' changes the plane $y = -\mu z$ into the plane $z = -\mu x$. The left member of the third equation under (1) remains invariant under Q_3'. The right changes but it is changed only by the equations $\begin{Bmatrix} x = y' \\ y = z' \\ z = x' \end{Bmatrix}$ which define T_3. Therefore, change the sign of the right member of the quadratic equation given; as,

$$-\{(1+\mu^2)x^2 - (1+\mu^4)z^2\}.$$

This corresponds to the line a_1 itself. Operate on this expression by T_3 and obtain

$$-\{(1+\mu^2)y^2 - (1+\mu^4)x^2\};$$

the latter corresponds to the line a_2. Hence, we conclude that a_1 goes into a_2.

In applying this process we find that Q_3' transforms $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0 into $a_2, a_3, a_1, a_5, a_6, a_4$ and the point 0, respectively.

The images of $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0 in the plane $w = 0$ are the fundamental points 1, 2, 3, 4, 5, 6, 7.* The images of the b_i lines are the conics $[i\,7]$. The image of the residual of b_i is the line $(i, 7)$. The image of the line c_{ik} is the line (i, k). The image of its residual is the conic $[i\,k]$. The image of the residuals of 0, a_i are the cubics [7] $[i]$.

Hence, the corresponding Cremona transformation is

$$T_3' \equiv \begin{pmatrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ 2, & 3, & 1, & 5, & 6, & 4, & 7 \end{pmatrix}.$$

This notation means that the image of 1 is 2, and so on for the other fundamental points. Since a transformation is fixed when the images of its fundamental points are known, we have our required transformation which is linear and of period 3.

If we had taken the transformation Q_3'' which also corresponds to T_3, we would have been led to a new transformation T_3''. This can also be found by

* The symbol (ij) denotes the line ij, $[ij]$ the conic through the remaining five fundamental points, and $[i]$ the cubic with a double point at i which passes through the other six fundamental points.

operating on T_3' by θ_2, the Geiser transformation. It transforms the point 1 into the residual of its image in the transformation T_3'. Similarly for the other fundamental points. Therefore,

$$T_3''\equiv\begin{Bmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ [2], & [3], & [1], & [5], & [6], & [4], & [7]\end{Bmatrix}$$

which is of order 8.

The transformation $T_2\equiv\begin{Bmatrix}x, y, z\\ -x', y', z'\end{Bmatrix}$ leads to the transformation

$$T_2'\equiv\begin{Bmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ 1, & [13], & [12], & [17], & [16], & [15], & [14]\end{Bmatrix}.$$

The transformation $T_2''\equiv T_2'\cdot\theta_2$ is

$$T_2''\equiv\begin{Bmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ [1], & (13), & (12), & (17), & (16), & (15), & (14)\end{Bmatrix}$$

which is a Jonquières transformation* of order 4.

Use the same fundamental process on $\overline{T}_2\equiv\begin{pmatrix}x, y, z\\ y, x, z\end{pmatrix}$. We are led to

$$\overline{T}_2'\equiv\begin{Bmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ [57], & [47], & [67], & [27], & [17], & [37], & [7]\end{Bmatrix}$$

which is of order 5 and

$$\overline{T}_2''\equiv\begin{Bmatrix}1, & 2, & 3, & 4, & 5, & 6, & 7\\ (57), & (47), & (67), & (27), & (17), & (37), & [7]\end{Bmatrix}$$

which is a Jonquières transformation of order 4.

G_{48} *with Some of its Subgroups.*

13. Using the notation of Section 6 we can arrive at different groups:

$$G_4\equiv[T_2', \theta_2]\,(1,1)\,[Q_2', \Gamma]\,(2,1)\,[T_2]\equiv\overline{G}_2 \text{ on } C_4,$$
$$G_6\equiv[T_3', \theta_2]\,(1,1)\,[Q_3', \Gamma]\,(2,1)\,[T_3]\equiv\overline{G}_3 \text{ on } C_4,$$
$$G_4\equiv[\overline{T}_2', \theta_2]\,(1,1)\,[\overline{Q}_2', \Gamma]\,(2,1)\,[T_2']\equiv\overline{G}_2 \text{ on } C_4,$$
$$G_{12}\equiv[T_3', T_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \Gamma]\,(2,1)\,[T_3, T_2]\equiv\overline{G}_6 \text{ on } C_4,$$
$$G_{16}\equiv[\overline{T}_2', T_2', \theta_2]\,(1,1)\,[\overline{Q}_2', Q_2', \Gamma]\,(2,1)\,[T_2, T_2]\equiv\overline{G} \text{ on}_8\ C_4,$$
$$G_{24}\equiv[T_3', T_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \Gamma]\,(2,1)\,[T_3, T_2]\equiv\overline{G}_{12} \text{ on } C_4,$$
$$G_{48}\equiv[T_3', T_2', \overline{T}_2', \theta_2]\,(1,1)\,[Q_3', Q_2', \overline{Q}_2', \Gamma]\,(2,1)\,[T_3, T_2, T_2]\equiv\overline{G}_{24}.$$

On account of the isomorphism existing between the groups G_{2n} and the groups $\overline{G}_n$ many other groups may be found such that groups G_m are simply isomorphic to $\overline{G}_m$.

* K. Doehlemann, "Geometrische Transformationen," Teil II (1907), p. 150.

14. Since the value of μ in our previous discussion may have any finite value, we may expect to find groups of higher order than 48 by restricting the value of μ.

Determination of G_{192}.

15. Let $k=0$, hence from the equation $\frac{4\mu^2}{(1+\mu^2)^2}-2=0$, we deduce

$$\mu^4+1=0. \tag{27}$$

Therefore, the quartic curve (13) has an equation of the form

$$\left\{\begin{matrix} x^4+y^4+z^4=0 \\ u \quad\quad\quad =0 \end{matrix}\right\}, \tag{28}$$

which has been studied by Dyck.* This curve is invariant under the group $\overline{G}_{96}$ of order 96. The octahedral group $\overline{G}_{24}$ and the transformation $T_4 \equiv \begin{pmatrix} x, & y, & z \\ ix', & y', & z' \end{pmatrix}$ generate it. Hence, in order to determine the corresponding group G_{192} all that remains is to determine a transformation in the plane $w=0$ corresponding to T_4.

Repeat the argument precisely as given in the previous case understanding that μ is subject to the condition

$$\mu^4+1=0$$

in all of the previous equations.

We are led to the transformations T_4' of order 6,

$$\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [1], & [46], & [67], & [36], & (15), & [5], & [26] \end{matrix}\right\}$$

$$\text{and } T_4'' \equiv T_4' \cdot \theta_2 \equiv \left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ (1), & (46), & (67), & (36), & [15], & (5), & (26) \end{matrix}\right\}$$

is a Jonquière transformation of order 3.

T_4' and T_4'' correspond to $T_4 \equiv \left\{\begin{matrix} x, & y, & z \\ ix', & y', & z' \end{matrix}\right\}$.

Hence, the group $G_{192} \equiv [G_{24}, T_4', \theta_2]$ (2,1) the group $\overline{G}_{96} \equiv [\overline{G}_{24}, T_4]$.

Determination of G_{336}.

16. Consider the case where μ satisfies the equation

$$\mu^2-\mu+2=0. \tag{29}$$

The value of k subject to this restriction is -3μ. Hence, the quartic C_4 reduces to the form

$$C_4 \equiv \left\{\begin{matrix} x^4+y^4+z^4-3\mu(x^2y^2+x^2z^2+y^2z^2)=0 \\ u \quad\quad\quad\quad\quad\quad\quad\quad =0 \end{matrix}\right\}. \tag{30}$$

* W. Dyck, "Notiz über eine reguläre Riemannische Fläche vom Geschlechte drei und die zugehörige Normalcurve vierter Ordnung," *Mathematische Annalen*, Vol. XVIII (1881), pp. 510–516; see p. 512.

Ciani* states that this quartic, subject to the condition (29), can be reduced by a suitable linear transformation to the form

$$\left\{\begin{matrix} x^3 z+y^3 x+z^3 y=0 \\ u \quad =0 \end{matrix}\right\}. \tag{31}$$

Klein† has shown that the latter admits a group of collineations $\overline{G}_{168}$ which is generated by an octahedral group $\overline{G}_{24}$ and a certain cyclic group $\overline{G}_7$.

Hence, the quartic in the form given by Ciani is invariant under a group $\overline{G}_{168}$ which is generated by our octahedral group $\overline{G}_{24}$ and some $\overline{T}_7$. Such a transformation

$$T_7 \equiv \left\{\begin{matrix} x=-[2x'-\mu(y'-z')] \\ y=2x'+\mu(y'-z') \\ z=\mu^2(y'+z') \end{matrix}\right\}$$

is given by Sharpe. ‡

Hence, we repeat the argument which is given for G_{48} with the restriction that μ satisfies the equation

$$\mu^2-\mu+2=0$$

in order to determine a transformation T'_7 corresponding to T_7. Since some care is necessary in determining the images of $a_1, a_2, a_3, a_4, a_5, a_6$ and the point 0, the details in finding the image of a_1 are given.

First operate on the quartic

$$\left\{\begin{matrix} x^4+y^4+z^4-3\mu(x^2y^2+x^2z+y^2z^2)=0 \\ u \quad =0 \end{matrix}\right\}$$

with the transformation T_7. Select the special term containing x^4 which we find to be

$$16(2+k)x^4.$$

Since $$k=\frac{4\mu^2}{(1+\mu^2)^2}-2,$$

therefore, $$16(2+k)x^4=\frac{(16)(4)\mu^2}{(1+\mu^2)^2}\cdot x^4.$$

Hence, in extracting the root we find the factor

$$\sqrt{16(2+k)}=\pm\frac{8\mu}{1+\mu^2},$$

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* E. Ciani, "I Varii Tipi Possibili di Quartiche Piane più Volte Omologico Armoniche," *Rendiconti del circolo Matematico di Palermo,* Vol. XIII (1899), pp. 347–373; see p. 365.

† F. Klein, "Ueber die Transformation siebenter Ordnung der Elliptischen Functionen, *Mathematische Annalen,* Vol. XIV (1879), pp. 428–471; see p. 446.

‡ F. R. Sharpe, "Conics through Inflections of Self-Projective Quartics," American Journal of Mathematics, Vol. XXXVII (1915), pp. 66–72; see p. 71.

but since $\mu^2-\mu+2=0$,

$$\sqrt{16(2+k)}=-\frac{8\mu}{1+\mu^2}=4\mu^2.$$

Hence, in making certain comparisons the factor $4\mu^2$ must be taken into account.

We now operate with the Q'_7 corresponding to T_7 as follows: The line a_1 is determined by

$$\begin{gathered} y=-\mu z \\ (x+y+z)v+\mu(x^2+y^2+z^2)=-\{(1+\mu^2)x^2-(1+\mu^4)z^2\}, \end{gathered}$$

or

$$\begin{gathered} y=-\mu z \\ (x+y+z)v+\mu(x^2+y^2+z^2)=-(\mu-1)\{x^2+3z^2\}. \end{gathered}$$

Operate on $y=-\mu z$ *by* T_7 and we get the equation

$$z=\frac{\mu^2}{4}(y-x). \tag{32}$$

Hence, a_1 goes into the line a_6 or its residual. To decide which, operate on

$$-(\mu-1)\{x^2+3z^2\} \tag{33}$$

by

$$x=-[2x'-\mu(y'-z')], \quad z=\frac{\mu^2}{4}(y'+z'), \tag{34}$$

where $z'=\dfrac{\mu^2}{4}(y''-x'')$ as in equation (32).

The coefficient of x''^2 reduces to $-\frac{1}{2}\mu^7$, but a_6 is defined by

$$z=\frac{\mu^2}{4}(y-x),(x+y+z)v+\mu(x^2+y^2+z^2)=\frac{+2\mu^5}{16}\left\{y^2+x^2-\frac{2\lambda^2+1}{\lambda^2}\cdot xy\right\}. \tag{35}$$

Compare the x''^2 term with the corresponding term in (35) remembering the former contains an extra factor $4\mu^2$, due to operating with T_7. Hence, we conclude that a_1 goes into the residual of a_6. Similarly, we show that a_3 and 0 go into the lines c_{36}, c_{56} while a_2, a_4, a_5, a_6 go into the residuals of c_{12}, b_6, c_{24}, c_{14}, respectively. Hence, in the w plane we get the transformations T'_7 of order 5,

$$\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [6], & [12], & (36), & (67), & [24], & [14], & (56) \end{matrix}\right\}$$

and

$$T'_7\cdot\theta_2\equiv T''_7\equiv\left\{\begin{matrix} 1, & 2, & 3, & 4, & 5, & 6, & 7 \\ [6], & (12), & [36], & [67], & (24), & (14), & [56] \end{matrix}\right\}.$$

Hence, the group $C_{336}\equiv[G_{24}, T'_7, \theta_2](2,1)$, the group $\overline{G}_{168}\equiv[\overline{G}_{24}, T_7]$.

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